

On Domination Number and Distance in Graphs

Cong X. Kang

Texas A&M University at Galveston, Galveston, TX 77553, USA

kangc@tamug.edu

September 16, 2014

Abstract

A vertex set S of a graph G is a *dominating set* if each vertex of G either belongs to S or is adjacent to a vertex in S . The *domination number* $\gamma(G)$ of G is the minimum cardinality of S as S varies over all dominating sets of G . It is known that $\gamma(G) \geq \frac{1}{3}(\text{diam}(G) + 1)$, where $\text{diam}(G)$ denotes the diameter of G . Define C_r as the largest constant such that $\gamma(G) \geq C_r \sum_{1 \leq i < j \leq r} d(x_i, x_j)$ for any r vertices of an arbitrary connected graph G ; then $C_2 = \frac{1}{3}$ in this view. The main result of this paper is that $C_r = \frac{1}{r(r-1)}$ for $r \geq 3$. It immediately follows that $\gamma(G) \geq \mu(G) = \frac{1}{n(n-1)}W(G)$, where $\mu(G)$ and $W(G)$ are respectively the average distance and the Wiener index of G of order n . As an application of our main result, we prove a conjecture of DeLaViña et al. that $\gamma(G) \geq \frac{1}{2}(\text{ecc}_G(B) + 1)$, where $\text{ecc}_G(B)$ denotes the eccentricity of the boundary of an arbitrary connected graph G .

Key Words: domination number, distance, diameter, spanning tree

2010 Mathematics Subject Classification: 05C69, 05C12

1 Introduction

We consider finite, simple, undirected, and connected graphs $G = (V(G), E(G))$ of order $|V(G)| \geq 2$ and size $|E(G)|$. For $W \subseteq V(G)$, we denote by $\langle W \rangle_G$ the subgraph of G induced by W . For $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. Further, let $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$ for $S \subseteq V(G)$. The degree of a vertex $v \in V(G)$ is $\deg_G(v) = |N_G(v)|$. The *distance* between two vertices $x, y \in V(G)$ in the subgraph H , denoted by $d_H(x, y)$, is the length of the shortest path between x and y in the subgraph H . The *diameter* $\text{diam}(H)$ of a graph H is $\max\{d_H(x, y) \mid x, y \in V(H)\}$.

A set $S \subseteq V(G)$ is a *dominating set* (resp. *total dominating set*) of G if $N[S] = V(G)$ (resp. $N(S) = V(G)$). The *domination number* (resp. *total domination number*) of G , denoted by $\gamma(G)$ (resp. $\gamma_t(G)$), is the minimum cardinality of S as S varies over all dominating sets (resp. total dominating sets) in G ; a dominating set (resp. total dominating set) of G of minimum cardinality is called a $\gamma(G)$ -set (resp. $\gamma_t(G)$ -set).

Both distance and (total) domination are very well-studied concepts in graph theory. For a survey of the myriad variations on the notion of domination in graphs, see [4].

It is well-known that $\gamma(G) \geq \frac{1}{3}(\text{diam}(G) + 1)$ (*); a “proof” to (*) can be found on p.56 of the authoritative reference [4]. However, the “proof” contained therein is logically flawed. We provide a counterexample to a crucial assertion in the “proof” and then present a correct proof to (*). Upon some

reflection, we see that (*) is the two parameter case of a family of inequalities existing between $\gamma(G)$ and the distances in G , in the following way: $\gamma(G) \geq \frac{1}{3}(\text{diam}(G) + 1) = \frac{1}{3\binom{r}{2}} \left(\binom{r}{2} \text{diam}(G) + \binom{r}{2} \right) \geq \frac{1}{3\binom{r}{2}} \left(\sum_{1 \leq i < j \leq r} d(x_i, x_j) \right)$. The inequality $\gamma(G) \geq \frac{1}{3\binom{r}{2}} \left(\sum_{1 \leq i < j \leq r} d(x_i, x_j) \right)$ naturally brings up the question: What is the largest constant C_r , such that $\gamma(G) \geq C_r \left(\sum_{1 \leq i < j \leq r} d(x_i, x_j) \right)$, for all connected graphs $G = (V, E)$ and arbitrary vertices $x_1, \dots, x_r \in V$, where $r \geq 2$? Taking this viewpoint, we have $C_2 = \frac{1}{3}$ by (*).

The main result of this paper is that $C_r = \frac{1}{r(r-1)}$ for $r \geq 3$. Since, for a graph G of order n , $W(G) = \sum_{1 \leq i < j \leq n} d(x_i, x_j)$ is the *Wiener index* of G (see [6]) and $\mu(G) = \frac{1}{n(n-1)} W(G)$ is the average distance (per definition found in [1]), it follows that $\gamma(G) \geq \mu(G) = \frac{1}{n(n-1)} W(G)$. As an application of our main result, we prove a conjecture in [3] by DeLaViña et al. that $\gamma(G) \geq \frac{1}{2}(\text{ecc}_G(B) + 1)$, where $\text{ecc}_G(B)$ denotes the eccentricity of the boundary of an arbitrary connected graph G (to be defined in Section 4).

This paper is motivated by the work of Henning and Yeo in [5], where they obtained similar inequalities for total domination number γ_t (rather than domination number γ). Given the close relation between the two graph parameters, we expect the techniques used in [5] to be readily adaptable towards the results of this paper. However, in striking contrast to [5], we avoid the painstaking case-by-case, structural analysis employed there by making use of the easy and well-known Lemma 3.1; this results in a much simpler and shorter paper. Further, we are able to obtain (in domination) the exact value of C_r for every r , rather than only a bound (in total domination, c.f. [5]) for C_r for all but the first few values of r .

2 An Error in the proof of $\gamma(G) \geq \frac{1}{3}(\text{diam}(G) + 1)$ in FoDiG

For readers' convenience, we first reproduce Theorem 2.24 and its incorrect proof as it appears on p.56 of [4], the authoritative reference in the field of domination titled *Fundamentals of Domination in Graphs*.

Theorem 2.1. *For any connected graph G , $\left\lceil \frac{\text{diam}(G)+1}{3} \right\rceil \leq \gamma(G)$.*

“Proof” (as found on p.56 of [4]). Let S be a γ -set of a connected graph G . Consider an arbitrary path of length $\text{diam}(G)$. This diametral path includes at most two edges from the induced subgraph $\langle N[v] \rangle$ for each $v \in S$. Furthermore, since S is a γ -set, the diametral path includes at most $\gamma(G) - 1$ edges joining the neighborhoods of the vertices of S . Hence, $\text{diam}(G) \leq 2\gamma(G) + \gamma(G) - 1 = 3\gamma(G) - 1$ and the desired result follows.” \square

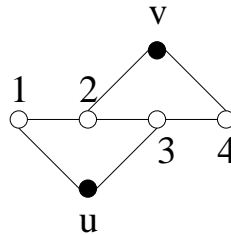


Figure 1: a counter-example

Presumably, by a “diametral path”, the authors had in mind an induced path with length $\text{diam}(G)$. Still, the assertion of the sentence beginning with “Furthermore” is incorrect, as seen by the example

in Figure 1: notice that $S = \{u, v\}$ is a γ -set and the vertices 1, 2, 3, 4 form a diametral path containing 3 edges joining $\langle N[u] \rangle$ with $\langle N[v] \rangle$, whereas $\gamma(G) - 1 = 1$.

3 Domination number and distance in graphs

The following lemma can be proved by exactly the same argument given in the proof of Lemma 2 in [2]; it was also observed on p.23 of [1].

Lemma 3.1. [1, 2] *Let M be a $\gamma(G)$ -set. Then there is a spanning tree T of G such that M is a $\gamma(T)$ -set.*

Now, we apply Lemma 3.1 to give a correct proof of Theorem 2.1.

Proof of Theorem 2.1. Given G , take a spanning tree T of G such that $\gamma(G) = \gamma(T)$. Suppose, for the sake of contradiction, $\gamma(G) < \frac{1}{3}(\text{diam}(G) + 1)$. Since $\gamma(T) = \gamma(G)$ and $\text{diam}(T) \geq \text{diam}(G)$, we have

$$\gamma(T) < \frac{1}{3}(\text{diam}(T) + 1) \quad (1)$$

Take a path P of T with length equal to $\text{diam}(T)$. If (1) holds, there must exist a vertex u of T such that $|V(P) \cap N[u]| \geq 4$. Since P is a path of T (a tree), this is impossible. \square

Theorem 3.2. *Given any three vertices x_1, x_2, x_3 of a connected graph G , we have*

$$\gamma(G) \geq \frac{1}{6}(d_G(x_1, x_2) + d_G(x_1, x_3) + d_G(x_2, x_3)). \quad (2)$$

Further, if equality is attained in (2), then $d_G(u, v) \equiv 2 \pmod{3}$ for any pair $u, v \in \{x_1, x_2, x_3\}$.

Proof. By Lemma 3.1, there exists a spanning tree T of G with $\gamma(T) = \gamma(G)$. Since $d_T(u, v) \geq d_G(u, v)$ for any two vertices $u, v \in V(T) = V(G)$, it suffices to prove (2) on T . If x_1, x_2 , and x_3 all lie on one geodesic, then the inequality (3) obviously holds by Theorem 2.1. Thus, let $d_T(x_1, y) = a$, $d_T(x_2, y) = b$, and $d_T(x_3, y) = c$, with $0 \notin \{a, b, c\}$, as shown in Figure 2. Then, the inequality (2) on T becomes

$$\gamma(T) \geq \frac{1}{6}((a + b) + (a + c) + (b + c)) = \frac{1}{3}(a + b + c). \quad (3)$$

Let y' be the vertex lying on the x_2 - y path and adjacent to y . Let P^1 and P^2 denote the x_1 - x_3 path and the x_2 - y' path, respectively. If there exists a $\gamma(T)$ -set M Not containing y , then M must contain a neighbor z of y . Suppose, WLOG, $z \neq y'$. Then, inequality (3) follows immediately from applying Theorem 2.1 to P^1 and P^2 . If y belongs to every $\gamma(T)$ -set M , then $\gamma(T) \geq 1 + \frac{1}{3}(a - 1) + \frac{1}{3}(b - 1) + \frac{1}{3}(c - 1) = \frac{1}{3}(a + b + c)$, and (3) again follows.

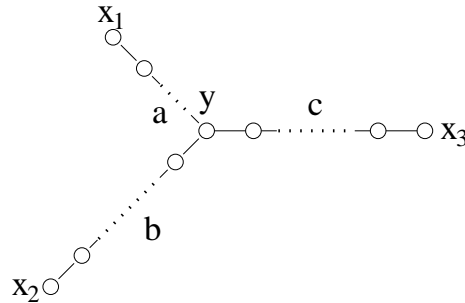


Figure 2: $r = 3$ case

Next, suppose equality is attained in (2). Again, let T be a spanning tree with $\gamma(T) = \gamma(G)$. Since $d_G(x_i, x_j) \leq d_T(x_i, x_j)$ and $\gamma(T) \geq \frac{1}{6}(d_T(x_1, x_2) + d_T(x_1, x_3) + d_T(x_2, x_3))$ holds, we have $\gamma(G) = \frac{1}{6}(d_G(x_1, x_2) + d_G(x_1, x_3) + d_G(x_2, x_3)) \leq \frac{1}{6}(d_T(x_1, x_2) + d_T(x_1, x_3) + d_T(x_2, x_3)) \leq \gamma(T)$. Thus, we deduce that $\gamma(T) = \frac{1}{6}(d_T(x_1, x_2) + d_T(x_1, x_3) + d_T(x_2, x_3))$ and $d_G(x_i, x_j) = d_T(x_i, x_j)$ for each pair (x_i, x_j) . With a, b, c defined as above, the present assumption implies $\gamma(T) = \frac{1}{3}(a + b + c)$. Observe, in light of Theorem 2.1, that the equality $\gamma(T) = \frac{1}{3}(a + b + c)$ is only possible if the following “optimal domination” of T occurs: there is a $\gamma(T)$ -set M which contains y , a degree-three vertex in $\langle V(P^1) \cup V(P^2) \rangle_T$ which dominates four or more vertices in T ; every other vertex of M dominates three or more vertices in T ; no vertex of T is dominated by more than one vertex of M . (Note that Figure 2 only shows $\langle V(P^1) \cup V(P^2) \rangle_T$, which may be a strict subgraph of T .) This “optimal domination” condition clearly implies that each member of $\{a, b, c\}$ must equal 1 (mod 3), which yields our second claim. \square

Next, we determine the largest C_r for $r \geq 3$ with the method deployed in [5]. However, rather than just getting a bound on C_r in the case of total domination there, we obtain the exact value of C_r for every r .

Theorem 3.3. *For $r \geq 3$, $C_r = \frac{1}{r^2 - r}$.*

Proof. First, we prove $C_r \leq \frac{1}{r^2 - r}$. Let $G = K_{1,r}$ be a star with r leaves labeled x_1, \dots, x_r . Then $\gamma(G) = 1$ and

$$\sum_{1 \leq i < j \leq r} d(x_i, x_j) = \binom{r}{2} \cdot 2 = r(r-1).$$

So, $C_r \leq \frac{1}{r(r-1)}$.

Next, we show that $C_r \geq \frac{1}{r^2 - r}$. Notice that $C_3 = \frac{1}{3(3-1)} = \frac{1}{6}$ is given by Theorem 3.2. Thus, let x_1, x_2, \dots, x_r be any arbitrary $r \geq 4$ vertices of G . Since $\gamma(G) \geq \frac{1}{6}(d_G(x_i, x_j) + d_G(x_i, x_k) + d_G(x_j, x_k))$ holds for any triplet $\{x_i, x_j, x_k\} \subseteq \{x_1, x_2, \dots, x_r\}$, we have

$$\binom{r}{3} \gamma(G) \geq \sum_{1 \leq i < j < k \leq r} \frac{1}{6}(d_G(x_i, x_j) + d_G(x_i, x_k) + d_G(x_j, x_k)) = \frac{r-2}{6} \sum_{1 \leq i < j \leq r} d(x_i, x_j);$$

note that the last equality comes from the fact that there are $r-2$ triplets containing any given pair of vertices. Thus, $C_r \geq \frac{1}{\binom{r}{3}} \frac{r-2}{6} = \frac{1}{r(r-1)}$ as well. \square

4 Applying Theorem 3.2 to a Conjecture of DeLaViña et al.

We need a few more definitions. The *eccentricity* of a vertex v in G , denoted by $\text{ecc}_G(v)$, is $\max\{d_G(v, x) \mid x \in V(G)\}$. The *boundary* of G is defined as the set $B(G) = \{v \in V(G) \mid \text{ecc}_G(v) = \text{diam}(G)\}$; we denote it simply as B hereafter. The distance between a vertex $v \in V(G)$ and a set $S \subseteq V(G)$ is defined as $d_G(v, S) = \min\{d_G(v, x) \mid x \in S\}$. Further, the eccentricity of $S \subseteq V(G)$ is defined as $\text{ecc}_G(S) = \max\{d_G(x, S) \mid x \in V(G)\}$.

In [3], DeLaViña et al. proved, for a tree G , that $\gamma(G) \geq \frac{1}{2}(\text{ecc}_G(B) + 1)$. They further conjectured that the inequality holds for any connected graph G . As an application of Theorem 3.2, we prove this conjecture. Our proof follows the arguments given by Henning and Yeo in [5] proving the analogous Graffiti.pc conjecture $\gamma_t(G) \geq \frac{2}{3}(\text{ecc}_G(B) + 1)$.

Theorem 4.1. *Let G be a connected graph. Then $\gamma(G) \geq \frac{1}{2}(\text{ecc}_G(B) + 1)$.*

Proof. If $B = V(G)$, then $\text{ecc}_G(B) = 0$ and the desired inequality obviously holds. So, suppose $B \neq V(G)$; this implies that $|V(G)| \geq 3$ and $|B| \geq 2$. Pick vertices x and y with $d(x, y) = \text{diam}(G)$; then, $x, y \in B$. Let $\text{ecc}_G(B) = R$. Pick $z \in V(G) \setminus B$ such that $d(z, B) = R$. We have $d(x, z) \geq R$, $d(y, z) \geq R$ and $d(x, y) = \text{diam}(G) \geq R + 1$. Hence, we have $d(x, y) + d(x, z) + d(y, z) \geq 3R + 1$ (\spadesuit). If equality holds in (\spadesuit), then $R = d(x, z) = d(y, z) = d(x, y) - 1$, and we can not have both $d(x, z)$ and $d(x, y)$ be congruent to 2 mod 3. In this case, by Theorem 3.2, we have that $\gamma(G) > \frac{1}{6}(d(x, y) + d(x, z) + d(y, z)) = \frac{1}{6}(3R + 1) = \frac{1}{2}R + \frac{1}{6}$, which implies $\gamma(G) \geq \frac{1}{2}R + \frac{1}{2}$. On the other hand, if the inequality (\spadesuit) is strict, again by Theorem 3.2, we have that $\gamma(G) \geq \frac{1}{6}(d(x, y) + d(x, z) + d(y, z)) > \frac{1}{6}(3R + 1) = \frac{1}{2}R + \frac{1}{6}$, which again implies $\gamma(G) \geq \frac{1}{2}R + \frac{1}{2}$. \square

References

- [1] P. Dankelmann, Average distance and domination number. *Discrete Appl. Math.* **80** (1997) 21-35.
- [2] E. DeLaViña, Q. Liu, R. Pepper, B. Waller, and D. West, Some conjectures of Graffiti.pc on total domination. *Cong. Numer.* 185 (2007), 81-95.
- [3] E. DeLaViña, R. Pepper, B. Waller, Lower Bounds for the Domination Number. *Discussiones Math. Graph Theory* **30**(3) (2010), 475-487
- [4] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York (1998).
- [5] M.A. Henning, A. Yeo, A new lower bound for the total domination number in graphs proving a Graffiti.pc Conjecture, *Discrete Appl. Math.*(2014), <http://dx.doi.org/10.1016/j.dam.2014.03.013>
- [6] H. Wiener, Structural determination of paraffin boiling points, *J. Amer. Chem. Soc.* 69 (1947) 17-20.